

HOCHSCHILD (CO)HOMOLOGY OF EXTERIOR ALGEBRAS USING ALGEBRAIC MORSE THEORY

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ABSTRACT. In [1], the authors computed the additive and multiplicative structure of $HH^*(A;A)$, where A is the n -th exterior algebra over a field. In this paper, we derive all their results using a different method (AMT), as well as calculate the additive structure of $HH_k(A;A)$ and $HH^k(A;A)$ over \mathbb{Z} . We provide concise presentations of algebras $HH_*(A;A)$ and $HH^*(A;A)$, as well as determine their generators in the Hochschild complex. Lastly, we compute an explicit free resolution (spanned by multisets) of the A^e -module A and describe the homotopy equivalence to its bar resolution.

Conventions. Throughout this article, R will denote a commutative unital ring, $A = \Lambda[x_1, \dots, x_n] = \Lambda(R^n)$ will be the n -th exterior algebra, and $A^e = A \otimes_R A^{\text{op}}$ its enveloping algebra, so that A - A -bimodules correspond to A^e -modules.

Symbols σ and τ will denote a set and a multiset respectively, or equivalently a strictly increasing and an increasing sequence, of elements from $[n] = \{1, \dots, n\}$. Let $\binom{[n]}{k} = \{k\text{-element subsets of } [n]\}$ and $\langle\langle [n] \rangle\rangle_k = \{k\text{-element multisubsets of } [n]\}$. Then we have $|\binom{[n]}{k}| = \binom{n}{k} = \frac{n!}{k!(n-k)!}$ and $|\langle\langle [n] \rangle\rangle_k| = \langle\langle n \rangle\rangle_k = \binom{n+k-1}{k}$.

For the sake of brevity, we omit the wedge \wedge and product \cdot symbols. We denote $x_\sigma = \bigwedge_{i \in \sigma} x_i$ and $x_\tau = \prod_{i \in \tau} x_i$, so that algebras $\Lambda[x_1, \dots, x_n]$ and $R[x_1, \dots, x_n]$ have R -module bases $\{x_\sigma; k \in \mathbb{N}, \sigma \in \binom{[n]}{k}\}$ and $\{x_\tau; k \in \mathbb{N}, \tau \in \langle\langle [n] \rangle\rangle_k\}$ respectively.

1. BAR RESOLUTION

Using algebraic Morse theory, we find an explicit homotopy equivalence between the bar resolution of bimodule A and a minimal free resolution.

1.1. Complex. The *bar resolution* of any associative unital R -algebra A is

$$\begin{aligned} B_*: \quad & \dots \longrightarrow A^{\otimes k+2} \longrightarrow A^{\otimes k+1} \longrightarrow \dots \longrightarrow A^{\otimes 2} \longrightarrow A \longrightarrow 0, \\ b_k: & a_0 \otimes \dots \otimes a_{k+1} \longmapsto \sum_{0 \leq i \leq k} (-1)^i a_0 \otimes \dots \otimes a_i a_{i+1} \otimes \dots \otimes a_{k+1}, \end{aligned}$$

which is an exact complex of A^e -modules, where $\otimes = \otimes_R$ and A^e acts on $A^{\otimes k+2}$ by

$$(\alpha \otimes \beta)(a_0 \otimes a_1 \otimes \dots \otimes a_k \otimes a_{k+1}) = (\alpha a_0) \otimes a_1 \otimes \dots \otimes a_k \otimes (a_{k+1} \beta).$$

Hochschild (co)homology of A with coefficients in an A^e -module M is the homology of the complex $(C_*, \partial_*) = M \otimes_{A^e} B_*$ and $(C^*, \delta^*) = \text{Hom}_{A^e}(B_*, M)$. Note that

$$M \otimes_{A^e} A^{\otimes k+2} \cong M \otimes_R A^{\otimes k}$$

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via $m \otimes (a_0 \otimes a_1 \otimes \dots \otimes a_k \otimes a_{k+1}) \mapsto (a_0 m a_{k+1}) \otimes (a_1 \otimes \dots \otimes a_k)$ and
 $\text{Hom}_{A^e}(A^{\otimes k+2}, M) \cong \text{Hom}_R(A^{\otimes k}, M)$
 via $\varphi \mapsto (a_1 \otimes \dots \otimes a_k \mapsto \varphi(1 \otimes a_1 \otimes \dots \otimes a_k \otimes 1))$.

1.2. AMT. To a chain complex of free modules (B_*, b_*) we associate a weighted digraph Γ_{B_*} (vertices are basis elements of B_* , weights of edges are nonzero entries of matrices b_*). Then we carefully select a matching \mathcal{M} in this digraph, so that its edges have invertible weights and if we reverse the direction of every $e \in \mathcal{M}$ in Γ_{B_*} , the obtained digraph $\Gamma_{B_*}^{\mathcal{M}}$ contains no directed cycles and no infinite paths in two adjacent degrees. Under these conditions, the AMT theorem (Sköldbberg [5], Welker & Jöllenbeck [2]) provides a homotopy equivalent complex $(\mathring{B}_*, \mathring{b}_*)$, spanned by the unmatched vertices in $\Gamma_{B_*}^{\mathcal{M}}$, and with the boundary \mathring{b}_* given by the sum of weights of directed paths in $\Gamma_{B_*}^{\mathcal{M}}$. For more details, we refer the reader to the two articles above (which specify the homotopy equivalence), or [3] for a quick formulation.

1.3. Digraph. Since R -module A is free on $\{x_\sigma; \sigma \subseteq [n]\}$, the A^e -module $A^{\otimes k+2}$ is free on $\{1 \otimes x_{\sigma_1} \otimes \dots \otimes x_{\sigma_k} \otimes 1; \sigma_1, \dots, \sigma_k \subseteq [n]\}$. These tensors are the vertices of the digraph Γ_{B_*} . From the definition of b_* , we see that the edges of Γ_{B_*} and their weights are of three different forms, namely

$$\begin{array}{ccccc}
 & & 1 \otimes x_{\sigma_1} \otimes \dots \otimes x_{\sigma_k} \otimes 1 & & \\
 & \swarrow x_{\sigma_1} \otimes 1 & \downarrow (-1)^i & \searrow (-1)^k 1 \otimes x_{\sigma_k} & \\
 1 \otimes x_{\sigma_2} \otimes \dots \otimes x_{\sigma_k} \otimes 1 & 1 \otimes x_{\sigma_1} \otimes \dots \otimes x_{\sigma_i} x_{\sigma_{i+1}} \otimes \dots \otimes x_{\sigma_k} \otimes 1 & & 1 \otimes x_{\sigma_1} \otimes \dots \otimes x_{\sigma_{k-1}} \otimes 1,
 \end{array}$$

where $x_{\sigma_i} x_{\sigma_{i+1}} = \begin{cases} (-1)^j x_{\sigma_i \cup \sigma_{i+1}}; & \sigma_i \cap \sigma_{i+1} = \emptyset \\ 0; & \text{otherwise} \end{cases}$ and j is the number of transpositions needed to transform the concatenated elements of σ_i, σ_{i+1} into an increasing order. By the normalized resolution [4, 1.1.14], we may assume all $\sigma_1, \dots, \sigma_k$ are nonempty.

1.4. Matching. Let us explain how we construct the matching. Given a vertex $1 \otimes x_{\sigma_1} \otimes \dots \otimes x_{\sigma_k} \otimes 1$, going up means splitting some σ_i into σ'_i and $\sigma_i \setminus \sigma'_i$. The simplest choice is $i = 1$ and $\sigma'_i = \{\max \sigma_i\}$, i.e. let $\mathcal{M} = \left\{ \begin{smallmatrix} 1 \otimes x_i \otimes x_{\sigma_1 \setminus \{i\}} \otimes \dots \otimes x_{\sigma_k} \otimes 1 \\ 1 \otimes x_{\sigma_1} \otimes \dots \otimes x_{\sigma_k} \otimes 1 \end{smallmatrix} ; i = \max \sigma_1 \right\}$. Then the unmatched vertices are $\mathring{\mathcal{M}} = \{1 \otimes x_{i_1} \otimes x_{\sigma_2} \otimes \dots \otimes x_{\sigma_k} \otimes 1; i_1 \leq \max \sigma_2\}$. If we add edges $\left\{ \begin{smallmatrix} 1 \otimes x_{i_1} \otimes x_i \otimes x_{\sigma_2 \setminus \{i\}} \otimes \dots \otimes x_{\sigma_k} \otimes 1 \\ 1 \otimes x_{i_1} \otimes x_{\sigma_2} \otimes \dots \otimes x_{\sigma_k} \otimes 1 \end{smallmatrix} ; i = \max \sigma_2 \right\}$ to \mathcal{M} , then the unmatched vertices are $\mathring{\mathcal{M}} = \{1 \otimes x_{i_1} \otimes x_{i_2} \otimes x_{\sigma_3} \otimes \dots \otimes x_{\sigma_k} \otimes 1; i_1 \leq i_2 \leq \max \sigma_3\}$. If we add edges $\left\{ \begin{smallmatrix} 1 \otimes x_{i_1} \otimes x_{i_2} \otimes x_i \otimes x_{\sigma_3 \setminus \{i\}} \otimes \dots \otimes x_{\sigma_k} \otimes 1 \\ 1 \otimes x_{i_1} \otimes x_{i_2} \otimes x_{\sigma_3} \otimes \dots \otimes x_{\sigma_k} \otimes 1 \end{smallmatrix} ; i = \max \sigma_3 \right\}$ to \mathcal{M} , then the unmatched vertices are $\mathring{\mathcal{M}} = \{1 \otimes x_{i_1} \otimes x_{i_2} \otimes x_{i_3} \otimes x_{\sigma_4} \otimes \dots \otimes x_{\sigma_k} \otimes 1; i_1 \leq i_2 \leq i_3 \leq \max \sigma_4\}$. Seeing the emerging pattern, we collectively define

$$\mathcal{M} = \left\{ \begin{smallmatrix} 1 \otimes x_{i_1} \otimes \dots \otimes x_{i_{r-1}} \otimes x_i \otimes x_{\sigma_r \setminus \{i\}} \otimes \dots \otimes x_{\sigma_k} \otimes 1 \\ 1 \otimes x_{i_1} \otimes \dots \otimes x_{i_{r-1}} \otimes x_{\sigma_r} \otimes \dots \otimes x_{\sigma_k} \otimes 1 \end{smallmatrix} ; r \geq 1, i_1 \leq \dots \leq i_{r-1} \leq i = \max \sigma_r \right\}.$$

For a multiset $\tau = \{i_1 \leq \dots \leq i_k\}$, we denote $x_{(\tau)} = 1 \otimes x_{i_1} \otimes \dots \otimes x_{i_k} \otimes 1$, and let $\bar{\tau}$ be its corresponding set, e.g. $\tau = \{1, 1, 2, 5, 5, 5\}$ implies $\bar{\tau} = \{1, 2, 5\}$.

Proposition 1. *The A^e -module A admits a resolution \mathring{B}_* , in which \mathring{B}_k is free on symbols $\{x_{(\tau)}; \tau \in \binom{[n]}{k}\}$ and $\mathring{b}_k(x_{(\tau)}) = \sum_{i \in \bar{\tau}} (x_i \otimes 1 + (-1)^k 1 \otimes x_i) x_{(\tau \setminus \{i\})}$.*

Since $x_i \otimes 1 \pm 1 \otimes x_i$ is a nonunit of A^e , this resolution is minimal.

Proof. By AMT, it suffices to show that: (i) \mathcal{M} is a Morse matching; (ii) determine its critical vertices and zig-zag paths.

(i) Any vertex can be written uniquely as $v = 1 \otimes x_{i_1} \otimes \dots \otimes x_{i_r} \otimes x_{\sigma_{r+1}} \otimes \dots \otimes x_{\sigma_k} \otimes 1$ where $i_1 \leq \dots \leq i_r$ and $r \geq 0$ is maximal. Then v is terminal iff $i_r \leq \max \sigma_{r+1}$, initial iff $i_r > \max \sigma_{r+1}$, and critical (=unmatched) iff $r = k$. Going up means separating the largest element from σ_{r+1} , going down means multiplying x_{i_r} and σ_{r+1} ; both moves are unique. This shows that \mathcal{M} is a matching. The weight of an edge in \mathcal{M} is ± 1 which is a unit of R . Since $A^{\otimes k+2}$ has a finite basis, the digraph $\Gamma_{B_*}^{\mathcal{M}}$ contains no infinite zig-zag paths with vertices in degree k and $k-1$. Suppose there exists a directed cycle in $\Gamma_{B_*}^{\mathcal{M}}$ with vertices in degree k and $k-1$, and along this cycle let

$$\begin{array}{ccccc} \dots & & 1 \otimes x_{i_1} \otimes \dots \otimes x_{i_r} \otimes x_{\sigma_{r+1}} \otimes \dots \otimes x_{\sigma_k} \otimes 1 & & \dots \\ \downarrow \mathcal{M} & \searrow & \downarrow \mathcal{M} & \searrow e & \downarrow \mathcal{M} \\ \dots & & 1 \otimes x_{i_1} \otimes \dots \otimes x_{i_r} \otimes x_{\sigma_{r+1}} \otimes \dots \otimes x_{\sigma_k} \otimes 1 & & \dots \end{array}$$

be the edge with largest r . What can the edge e be? Multiplying $1 \otimes x_{i_1}$ or $x_{\sigma_k} \otimes 1$ or $x_{\sigma_s} \otimes x_{\sigma_{s+1}}$ for $s \geq r+2$ does not lead to a terminal vertex. Multiplying $x_{\sigma_{r+1}} \otimes x_{\sigma_{r+2}}$ and going up via \mathcal{M} increases r , a contradiction with the maximality. Multiplying $x_{i_s} \otimes x_{i_{s+1}}$ and going up via \mathcal{M} gives $x_{i_{s+1}} \otimes x_{i_s}$. Here we obtained a decrease $i_{s+1} > i_s$ in the sequence, but every subsequent zig-zag always puts the largest element on the left, which means that we will never regain $x_{i_s} \otimes x_{i_{s+1}}$. This is a contradiction, hence $\Gamma_{B_*}^{\mathcal{M}}$ contains no directed cycles and \mathcal{M} is a Morse matching.

(ii) Critical vertices are $\mathring{\mathcal{M}} = \{1 \otimes x_{i_1} \otimes \dots \otimes x_{i_k} \otimes 1; i_1 \leq \dots \leq i_k\} = \{x_{(\tau)}\}$. From $x_{(\tau)}$, what are all the paths to all possible $x_{(\tau')}$? Every zig-zag replaces $x_{i_r} \otimes x_{i_{r+1}}$ by $x_{i_{r+1}} \otimes x_{i_r}$ where $i_r < i_{r+1}$ and it has weight $(-1)^{r+1+r+1} = 1$, but at the end of the zig-zag path, the sequence must be increasing. Thus after that, the only two options are: either keep moving x_{i_r} to the right until it reaches 1 and we obtain $(-1)^k 1 \otimes x_{i_r} \cdot x_{(\tau \setminus \{i_r\})}$, or keep moving $x_{i_{r+1}}$ to the left until it reaches 1 and we obtain $x_{i_{r+1}} \otimes 1 \cdot x_{(\tau \setminus \{i_{r+1}\})}$. The sequence $x_{(\tau)}$ may contain constant subsequences $x_i \otimes \dots \otimes x_i$, and in this case only the first x_i is moved to the beginning, and only the last x_i is moved to the end (because we must not multiply $x_i \otimes x_i$ since $x_i^2 = 0$). Thus the zig-zag paths from $x_{(\tau)}$ to $x_{(\tau')}$ correspond to elements of the set $\bar{\tau}$. \square

1.5. Homotopy equivalence. This subsection is only used for determining the cup products on $HH^*(A; A)$ and can be skipped at first reading.

For a multiset $\tau = \{i_1 \leq \dots \leq i_k\}$, let S_τ denote the group of all its bijections, which is a subgroup of the usual symmetric group S_k of all bijections of $[k]$. Given $\pi \in S_\tau$, let $x_{(\pi\tau)} = 1 \otimes x_{\pi i_1} \otimes \dots \otimes x_{\pi i_k} \otimes 1 \in A^{\otimes k+2}$ denote the permuted tensor.

Proposition 2. *The homotopy equivalence $h: \mathring{B}_* \rightarrow B_*$ induced by \mathcal{M} sends*

$$h: x_{(\tau)} \mapsto \sum_{\pi \in S_\tau} x_{(\pi\tau)} \quad \text{and} \quad h^{-1}: x_{(\pi\tau)} \mapsto \begin{cases} x_{(\tau)}; & \pi = \text{id} \\ 0; & \pi \neq \text{id} \end{cases}.$$

If $v = 1 \otimes x_{\sigma_1} \otimes \dots \otimes x_{\sigma_k} \otimes 1 \in B_k$ is not a tensor of variables (i.e. contains at least one x_σ with $|\sigma| \geq 2$), then determining $h^{-1}(v)$ is more complicated.

Proof. By AMT, h sends $x_{(\tau)} \in \mathring{B}_k$ to $\sum_v \alpha_v v$ where $\alpha_v \in A^e$ is the sum of weights of all directed paths in $\Gamma_{B_*}^{\mathcal{M}}$ from $x_{(\tau)}$ to v , and h^{-1} sends $v \in B_k$ to $\sum_\tau \alpha_\tau x_{(\tau)}$ where $\alpha_\tau \in A^e$ is the sum of weights of all directed paths in $\Gamma_{B_*}^{\mathcal{M}}$ from v to $x_{(\tau)}$.

Determining h^{-1} : Since $x_{(\tau)}$ is not an endpoint of any $e \in \mathcal{M}$, a path from $v \in B_*$ to $x_{(\tau)} \in \mathring{B}_*$ must be in degrees k and $k+1$. But if $v = x_{(\pi\tau')}$, then v is not terminal, so a path from v to $x_{(\tau)}$ exists iff $\tau = \tau'$ and $\pi = \text{id}$, namely the path of length 0.

Determining h : Since $x_{(\tau)}$ is not an endpoint of any $e \in \mathcal{M}$, a path from $x_{(\tau)}$ to v must be in degrees k and $k-1$. Every zig-zag out of $x_{(\tau)}$ switches two consecutive elements (i.e. is a transposition $\xi_i = (i, i+1)$ with weight 1), vertex v must be a permuted $x_{(\tau)}$. Thus it remains to prove: from any $x_{(\tau)}$ to any $x_{(\pi\tau)}$ there exists precisely one path (this is equivalent to the existence of a specific normal form on the presentation of S_τ by generators ξ_i). Notice that a path from $x_{(\tau)}$ to $x_{(\pi\tau)}$ does not correspond to an arbitrary product of transpositions, because if we switch $x_{i_r} \otimes x_{i_{r+1}}$, then switching $x_{i_s} \otimes x_{i_{s+1}}$ with $s \geq r+2$ is not possible, since it leads to a nonterminal vertex. For example, from $x_{(2,5,5,9)}$ to $x_{(9,5,5,2)}$, the only path is

$$\begin{array}{ccccccccc} x_{(\tau)} = x_{(2,5,5,9)} & & x_{(5,2,5,9)} & & x_{(5,5,2,9)} & & x_{(5,5,9,2)} & & x_{(5,9,5,2)} & & x_{(9,5,5,2)} = x_{(\pi\tau)} \\ \downarrow \swarrow \mathcal{M} & & \downarrow \swarrow \mathcal{M} & & \downarrow \swarrow \mathcal{M} & & \downarrow \swarrow \mathcal{M} & & \downarrow \swarrow \mathcal{M} & & \\ x_{(25,5,9)} & & x_{(5,25,9)} & & x_{(5,5,29)} & & x_{(5,59,2)} & & x_{(59,5,2)} & & \end{array}$$

In general, from $x_{(\tau)} = x_{(i_1, \dots, i_k)}$ to $x_{(\pi\tau)} = x_{(j_1, \dots, j_k)}$ there is the following path. First move j_k to position k (producing $x_{(i_1, \dots, \widehat{j_k}, \dots, i_k, j_k)}$) via $\prod_{i=\epsilon j_k}^{k-1} \xi_i =: \xi_{(j_k)}$, where ϵj_k is the position of the last j_k in i_1, \dots, i_k . Then move j_{k-1} to position $k-1$ (producing $x_{(i_1, \dots, \widehat{j_{k-1}}, \dots, \widehat{j_k}, \dots, i_k, j_{k-1}, j_k)}$) via $\prod_{i=\epsilon j_{k-1}}^{k-2} \xi_i =: \xi_{(j_{k-1})}$, where ϵj_{k-1} is the position of the last j_{k-1} in $i_1, \dots, \widehat{j_k}, \dots, i_k$. Then move j_{k-2} to position $k-2$ (producing $x_{(i_1, \dots, \widehat{j_{k-2}}, \dots, \widehat{j_{k-1}}, \dots, \widehat{j_k}, \dots, i_k, j_{k-2}, j_{k-1}, j_k)}$) via $\prod_{i=\epsilon j_{k-2}}^{k-3} \xi_i =: \xi_{(j_{k-2})}$, where ϵj_{k-2} is the position of the last j_{k-2} in $i_1, \dots, \widehat{j_{k-1}}, \dots, \widehat{j_k}, \dots, i_k$. Continuing this process, every j_r is moved to position r , so applying $\xi_{(j_1)} \cdots \xi_{(j_k)}$ to $x_{(\tau)}$ produces $x_{(\pi\tau)}$.

This product corresponds to a valid path, since in every $\xi_{(j_r)}$ when we move j_r to the right, left of j_r is an increasing sequence of i 's at each step. Every path corresponds to some $\xi_{(j_1)} \cdots \xi_{(j_k)}$, since the first step must be moving j_k to position k (if we first apply ξ_r with $r < \epsilon j_k$, then moving j_k to the right does not lead to a terminal vertex; if we first apply ξ_r with $r > \epsilon j_k$, then j_k will not be able to move to the right across the transposed i_{r+1}, i_r), and then inductively every next step must be moving j_r to position r . Any permutation $\pi \in S_\tau$ is equal to a unique product $\xi_{(j_1)} \cdots \xi_{(j_k)}$, since applying $\xi_{(j_1)} \cdots \xi_{(j_k)}$ to $x_{(\tau)}$ produces $x_{(j_1, \dots, j_k)}$. \square

2. HOMOLOGY

In the computation below, we implicitly use the isomorphisms from 1.1.

Theorem 3. *If $R = \mathbb{Z}$, then $HH_k(A; A) \cong \mathbb{Z}^F \oplus \mathbb{Z}_2^T$, where*

$$F = 2^{n-1} \binom{n}{k} + \begin{cases} 1; & k=0 \\ 0; & k \geq 1 \end{cases} \quad \text{and} \quad T = (-1)^{k+1} + 2^{n-1} \sum_{0 \leq i \leq k} (-1)^{k-i} \binom{n}{i}.$$

In particular, if $R = K$ is a field, then

$$\dim_K HH_k(A; A) = \begin{cases} 2^n \binom{n}{k}; & \text{char } K = 2 \\ 2^{n-1} \binom{n}{k}; & \text{char } K \neq 2, k \geq 1 \\ 2^{n-1} + 1; & \text{char } K \neq 2, k = 0 \end{cases}.$$

Proof. $A \otimes_{A^e} \mathring{B}_k$ has an R -module basis $\{x_\sigma \otimes x_{(\tau)}; \sigma \subseteq [n], \tau \in \binom{[n]}{k}\}$. The homotopy equivalence $B_* \simeq \mathring{B}_*$ implies $A \otimes_{A^e} B_* \simeq A \otimes_{A^e} \mathring{B}_* =: \mathring{C}_*$ in which the new boundary is

$$\begin{aligned} \mathring{\partial}_k(x_\sigma \otimes x_{(\tau)}) &= x_\sigma \otimes \mathring{b}_k(x_{(\tau)}) = x_\sigma \otimes \sum_{i \in \tau} (x_i \otimes 1 + (-1)^k 1 \otimes x_i) x_{(\tau \setminus \{i\})} = \\ &= \sum_{i \in \tau} (x_\sigma x_i + (-1)^k x_i x_\sigma) \otimes x_{(\tau \setminus \{i\})} = \sum_{i \in \tau} ((-1)^{|\sigma|} + (-1)^{|\tau|}) x_i x_\sigma \otimes x_{(\tau \setminus \{i\})}. \end{aligned}$$

The chain complex $(\mathring{C}_*, \mathring{\partial}_*)$ is a direct sum of subcomplexes

$$\begin{aligned} C'_* &= \langle x_\sigma \otimes x_{(\tau)}; (-1)^{|\sigma|} = (-1)^{|\tau|} \rangle, \quad \partial'(x_\sigma \otimes x_{(\tau)}) = \sum_{i \in \tau \setminus \sigma} (-1)^{|\sigma|+j} 2 x_{\sigma \cup \{i\}} \otimes x_{(\tau \setminus \{i\})}, \\ C''_* &= \langle x_\sigma \otimes x_{(\tau)}; (-1)^{|\sigma|} \neq (-1)^{|\tau|} \rangle, \quad \partial'' = 0, \quad \text{where } x_i x_\sigma = (-1)^j x_{\sigma \cup \{i\}}. \end{aligned}$$

Let $R = K$ be a field. If $\text{char } K = 2$, then $\mathring{\partial}_* = 0$ and thus $\dim_K HH_k(A; A) = |\{x_\sigma \otimes x_{(\tau)}\}| = 2^n \binom{n}{k}$. If $\text{char } K \neq 2$, then $(-1)^{|\sigma|} 2$ is a unit of K , so we define

$$\mathcal{M}' = \left\{ \begin{array}{c} x_\sigma \otimes x_{(\tau \cup \{i\})} \\ \downarrow \\ x_{\sigma \cup \{i\}} \otimes x_{(\tau)} \end{array} ; \max \sigma < i \leq \max \tau \right\}.$$

This is a Morse matching on C'_* with no zig-zags, and the critical vertices are $\mathcal{M}' = \{x_\emptyset \otimes x_{(\emptyset)}\}$. Hence C'_* and C''_* contribute $\begin{Bmatrix} 1; & k=0 \\ 0; & k \geq 1 \end{Bmatrix}$ and $2^{n-1} \binom{n}{k}$ to $\dim_K HH_k(A; A)$.

Let $R = \mathbb{Z}$, so ± 2 is no longer a unit. Instead of studying (C'_*, ∂') , let us observe $(C'_*, \frac{\partial'}{2})$. We will inductively use a trick: if matrices $\mathbb{Z}^l \xleftarrow{\alpha} \mathbb{Z}^m \xleftarrow{\beta} \mathbb{Z}^n$ satisfy $\alpha\beta = 0$ and have SNF (Smith normal forms) $\alpha \equiv \text{diag}(a_1, \dots, a_r, 0, \dots, 0)$ and $\beta \equiv \text{diag}(b_1, \dots, b_s, 0, \dots, 0)$, then $\frac{\text{Ker } \alpha}{\text{Im } \beta} \cong \mathbb{Z}^{m-r-s} \oplus \mathbb{Z}_{b_1} \oplus \dots \oplus \mathbb{Z}_{b_s}$ and $\frac{\text{Ker } 2\alpha}{\text{Im } 2\beta} \cong \mathbb{Z}^{m-r-s} \oplus \mathbb{Z}_{2b_1} \oplus \dots \oplus \mathbb{Z}_{2b_s}$. Since the above \mathcal{M}' is a Morse matching on $(C'_*, \frac{\partial'}{2})$, it follows that $H_k(C'_*, \frac{\partial'}{2}) \cong \begin{Bmatrix} \mathbb{Z}; & k=0 \\ 0; & k \geq 1 \end{Bmatrix}$. Hence boundaries $\frac{\partial'_k}{2}$ have SNFs

$$\mathbb{Z}^{r_0} \xleftarrow{\text{diag}(\overbrace{1, \dots, 1}^{r_0-1}, 0, \dots, 0)} \mathbb{Z}^{r_1} \xleftarrow{\text{diag}(\overbrace{1, \dots, 1}^{r_1-r_0+1}, 0, \dots, 0)} \mathbb{Z}^{r_2} \leftarrow \dots \leftarrow \mathbb{Z}^{r_k} \xleftarrow{\text{diag}(\overbrace{1, \dots, 1}^{\sum_{i=1}^k (-1)^{k-i} r_i}, 0, \dots, 0)} \mathbb{Z}^{r_{k+1}} \leftarrow \dots,$$

where $r_k = 2^{n-1} \binom{n}{k}$ and $r_{-1} = 1$. Then we conclude that $H_k(C'_*, \partial'_*) \cong \mathbb{Z}^F \oplus \mathbb{Z}_2^T$, where free rank is $F = \begin{Bmatrix} 1; & k=0 \\ 0; & k \geq 1 \end{Bmatrix}$ and 2-torsion rank is $\sum_{i=1}^k (-1)^{k-i} r_i$. \square

3. COHOMOLOGY

Using finite additivity of $\text{Hom}(-, -)$, we see that $\text{Hom}_{A^e}(B_k, A)$ has a basis $\{\varphi_{v,\sigma}; v \in \{1 \otimes x_{\sigma_1} \otimes \dots \otimes x_{\sigma_k} \otimes 1; \sigma_1, \dots, \sigma_k \subseteq [n]\}, \sigma \subseteq [n]\}$ and $\text{Hom}_{A^e}(\mathring{B}_k, A)$ has a basis $\{\varphi_{\tau,\sigma}; \tau \in \binom{[n]}{k}, \sigma \subseteq [n]\}$, the dual bases of B_k and \mathring{B}_k , where $\varphi_{v,\sigma}(v') = \begin{cases} x_\sigma; & v=v' \\ 0; & v \neq v' \end{cases}$.

Theorem 4. *If $R = \mathbb{Z}$, then $HH^k(A; A) \cong \mathbb{Z}^F \oplus \mathbb{Z}_2^T$, where*

$$F = 2^{n-1} \binom{n}{k} + \begin{cases} 1; & k=0 \text{ and } n \text{ odd} \\ 0; & \text{otherwise} \end{cases} \quad \text{and} \quad T = 2^{n-1} \sum_{0 \leq i < k} (-1)^{k-1-i} \binom{n}{i} + \begin{cases} (-1)^k; & n \text{ odd} \\ 0; & n \text{ even} \end{cases}.$$

In particular, if $R = K$ is a field, then

$$\dim_K HH^k(A; A) = \begin{cases} 2^n \binom{n}{k}; & \text{char } K = 2 \\ 2^{n-1} \binom{n}{k}; & \text{char } K \neq 2, (k \geq 1 \text{ or } n \text{ even}) \\ 2^{n-1} + 1; & \text{char } K \neq 2, (k=0 \text{ and } n \text{ odd}) \end{cases}.$$

Proof. Homotopy equivalence $B_* \simeq \mathring{B}_*$ implies $\text{Hom}_{A^e}(B_*, A) \simeq \text{Hom}_{A^e}(\mathring{B}_*, A) =: \mathring{C}^*$. In this complex, the new coboundary is

$$\begin{aligned} (\mathring{\delta}_k \varphi_{\tau, \sigma})(x_{(\tau)}) &= \varphi_{\tau, \sigma}(\mathring{b}_k(x_{(\tau)})) = \sum_{i \in \overline{\tau}'} (x_i \otimes 1 + (-1)^{|\tau'|} 1 \otimes x_i) \varphi_{\tau, \sigma}(x_{(\tau \setminus \{i\})}) = \\ &= \sum_{i \in \overline{\tau}'} (x_i \otimes 1 + (-1)^{|\tau'|} 1 \otimes x_i) \cdot \begin{cases} x_\sigma; & \tau' = \tau \cup \{i\} \\ 0; & \tau' \neq \tau \cup \{i\} \end{cases} = \sum_{i \in \overline{\tau}'} \begin{cases} ((-1)^{|\sigma|} + (-1)^{|\tau'|} x_\sigma x_i); & \tau' = \tau \cup \{i\} \\ 0; & \tau' \neq \tau \cup \{i\} \end{cases}, \\ \text{therefore } \mathring{\delta}_k(\varphi_{\tau, \sigma}) &= \sum_{i \in [n] \setminus \sigma} ((-1)^{|\sigma|} - (-1)^{|\tau|}) (-1)^j \varphi_{\tau \cup \{i\}, \sigma \cup \{i\}}, \end{aligned}$$

where j is the number of transpositions, needed to transform $x_\sigma x_i$ into $x_{\sigma \cup \{i\}}$. The cochain complex $(\mathring{C}^*, \mathring{\delta}^*)$ is a direct sum of subcomplexes

$$\begin{aligned} C'^* &= \langle \varphi_{\tau, \sigma}; (-1)^{|\sigma|} \neq (-1)^{|\tau|} \rangle, \quad \delta'(\varphi_{\tau, \sigma}) = \sum_{i \in [n] \setminus \sigma} (-1)^{|\sigma|+j} 2 \varphi_{\tau \cup \{i\}, \sigma \cup \{i\}}, \\ C''^* &= \langle \varphi_{\tau, \sigma}; (-1)^{|\sigma|} = (-1)^{|\tau|} \rangle, \quad \delta'' = 0. \end{aligned}$$

Let $R = K$ be a field. If $\text{char } K = 2$, then $\mathring{\delta}^* = 0$ and thus $\dim_K HH^k(A; A) = |\{\varphi_{\tau, \sigma}\}| = 2^n \binom{n}{k}$. If $\text{char } K \neq 2$, then $(-1)^{|\sigma|+j} 2$ is a unit of K , so we define

$$\mathcal{M}' = \left\{ \begin{array}{c} \varphi_{\tau \cup \{i\}, \sigma \cup \{i\}} \\ \downarrow \\ \varphi_{\tau, \sigma} \end{array} ; [n] \setminus \overline{\tau} \supseteq [i-1] \subseteq \sigma, i \notin \sigma \right\}.$$

This is a Morse matching on C'^* with critical vertices $\mathring{\mathcal{M}}' = \{\varphi_{\emptyset, [n]}; n \text{ odd}\}$. Hence C'^* and C''^* contribute $\begin{cases} 1; & k=0 \text{ and } n \text{ odd} \\ 0; & k \geq 1 \text{ or } n \text{ even} \end{cases}$ and $2^{n-1} \binom{n}{k}$ to $\dim_K HH^k(A; A)$.

Let $R = \mathbb{Z}$, so ± 2 is no longer a unit. Since \mathcal{M}' is a Morse matching on $(C'^*, \frac{\delta'^*}{2})$, it follows that $H^k(C'^*, \frac{\delta'^*}{2}) \cong \begin{cases} \mathbb{Z}; & k=0 \text{ and } n \text{ odd} \\ 0; & k \geq 1 \text{ or } n \text{ even} \end{cases}$. Hence coboundaries $\frac{\delta'^k}{2}$ have SNFs

$$\begin{aligned} \mathbb{Z}^{r_0} \xrightarrow{\text{diag}(1, \dots, 1, 0, \dots, 0)} \mathbb{Z}^{r_1} \xrightarrow{\text{diag}(1, \dots, 1, 0, \dots, 0)} \mathbb{Z}^{r_2} \longrightarrow \dots \longrightarrow \mathbb{Z}^{r_k} \xrightarrow{\text{diag}(1, \dots, 1, 0, \dots, 0)} \mathbb{Z}^{r_{k+1}} \text{ if } n \notin 2\mathbb{N}, \\ \mathbb{Z}^{r_0} \xrightarrow{\text{diag}(1, \dots, 1, 0, \dots, 0)} \mathbb{Z}^{r_1} \xrightarrow{\text{diag}(1, \dots, 1, 0, \dots, 0)} \mathbb{Z}^{r_2} \longrightarrow \dots \longrightarrow \mathbb{Z}^{r_k} \xrightarrow{\text{diag}(1, \dots, 1, 0, \dots, 0)} \mathbb{Z}^{r_{k+1}} \text{ if } n \in 2\mathbb{N}. \end{aligned}$$

Then we deduce that $H^k(C'^*, \delta'^*) \cong \mathbb{Z}^F \oplus \mathbb{Z}_2^T$, where free rank is $F = \begin{cases} 1; & k=0 \text{ and } n \notin 2\mathbb{N} \\ 0; & k \geq 1 \text{ or } n \in 2\mathbb{N} \end{cases}$ and 2-torsion rank is $T = \begin{cases} (-1)^k; & n \notin 2\mathbb{N} \\ 0; & n \in 2\mathbb{N} \end{cases} + \sum_{i=0}^{k-1} (-1)^{k-1-i} r_i$, with $r_k = 2^{n-1} \binom{n}{k}$. \square

4. CUP PRODUCT

For any associative unital algebra A , the cup product of $f \in \text{Hom}_R(A^{\otimes k}, A)$ and $g \in \text{Hom}_R(A^{\otimes l}, A)$ is $f \smile g \in \text{Hom}_R(A^{\otimes k+l}, A)$ given by

$$a_1 \otimes \dots \otimes a_k \otimes a_{k+1} \otimes \dots \otimes a_{k+l} \mapsto f(a_1 \otimes \dots \otimes a_k) \cdot g(a_{k+1} \otimes \dots \otimes a_{k+l})$$

and makes $HH^*(A; A)$ a graded-commutative R -algebra.

Theorem 5. *Let $R = K$ be a field. If $\text{char } K = 2$, then*

$$HH^*(A; A) \cong \Lambda[x_1, \dots, x_n] \otimes_K K[x_1, \dots, x_n],$$

where $x_\sigma \otimes x_\tau$ corresponds to $\varphi_{\tau, \sigma} \in \text{Hom}_R(A^{\otimes k}, A)$. If $\text{char } K \neq 2$, then $HH^(A; A)$ is isomorphic to the subalgebra of $\Lambda[x_1, \dots, x_n] \otimes_K K[x_1, \dots, x_n]$, spanned by*

$$\{x_\sigma \otimes x_\tau; (-1)^{|\sigma|} = (-1)^{|\tau|}\} \cup \{x_{[n]} \otimes 1\}.$$

Thus if $\text{char } K \neq 2$, then $HH^*(A; A)$ has algebra generators

$$\{1 \otimes x_{i,j}; i \leq j \in [n]\} \cup \{x_{i,j} \otimes 1; i < j \in [n]\} \cup \{x_i \otimes x_j; i, j \in [n]\} \cup \{x_{[n]} \otimes 1\},$$

and is subject to the relations [1, Table 1], though their result is missing $x_{[n]} \otimes 1$.

Proof. 1.5 induces a homotopy equivalence $\bar{h}: \text{Hom}_R(B_*, A) \rightarrow \text{Hom}_R(\check{B}_*, A)$, $\varphi \mapsto \varphi \circ h$, given by $\bar{h}(\varphi_{\pi\tau, \sigma}) = \varphi_{\tau, \sigma}$, $\bar{h}(\varphi_{v, \sigma}) = 0$, $\bar{h}^{-1}(\varphi_{\tau, \sigma}) = \varphi_{\tau, \sigma} + \sum_v \alpha_v \varphi_{v, \sigma}$, where v is not a tensor of variables. Indeed, $\varphi_{\pi\tau, \sigma} \circ h(x_{(\tau')}) = \varphi_{\pi\tau, \sigma} \sum_{\pi' \in S_{\tau'}} x_{(\pi' \tau')} = \begin{cases} x_{\sigma}; & \tau = \tau' \\ 0; & \tau \neq \tau' \end{cases}$, $\varphi_{v, \sigma} \circ h(x_{(\tau)}) = \varphi_{v, \sigma} \sum_{\pi \in S_{\tau}} x_{(\pi \tau)} = 0$, $\varphi_{\tau, \sigma} \circ h^{-1}(x_{(\pi \tau')}) = \begin{cases} x_{\sigma}; & \tau = \tau' \text{ and } \pi = \text{id} \\ 0; & \tau \neq \tau' \text{ or } \pi \neq \text{id} \end{cases}$. Then

$$\begin{aligned} \varphi_{\tau, \sigma} \smile \varphi_{\tau', \sigma'} &= \bar{h}(\bar{h}^{-1}(\varphi_{\tau, \sigma}) \smile \bar{h}^{-1}(\varphi_{\tau', \sigma'})) = \bar{h}((\varphi_{\tau, \sigma} + \dots) \smile (\varphi_{\tau', \sigma'} + \dots)) = \\ &= \bar{h}(\varphi_{\tau, \sigma} \smile \varphi_{\tau', \sigma'} + \dots) = \bar{h}((-1)^j \varphi_{\tau \otimes \tau', \sigma \sqcup \sigma'} + \dots) = (-1)^j \varphi_{\tau \cup \tau', \sigma \sqcup \sigma'}, \end{aligned}$$

where $(-1)^j$ is the sign of $x_{\sigma} x_{\sigma'}$. The three dots represent summands of the form $\varphi_{v, \sigma} \smile \varphi_{v', \sigma'} = (-1)^j \varphi_{v \otimes v', \sigma \sqcup \sigma'}$ where either v or v' is not a tensor of variables. \square

Remark 6. In general, $HH_*(A; A)$ and $HH^*(A; A)$ are modules over the center $Z(A)$. The multiplication map $A \otimes A \rightarrow A$, $x \otimes y \mapsto xy$ is an algebra morphism iff A is commutative, and then by [4, 4.2.6] it induces a *shuffle* product on $HH_*(A; A)$, given by $(a \otimes a_1 \otimes \dots \otimes a_i) \cdot (a' \otimes a_{i+1} \otimes \dots \otimes a_{i+j}) =$

$$\sum_{\pi \in S_{i+j}, \pi_1 < \dots < \pi_i, \pi_{i+1} < \dots < \pi_{i+j}} \text{sgn} \pi a a' \otimes a_{\pi_1^{-1}} \otimes \dots \otimes a_{\pi_i^{-1}} \otimes a_{\pi_{i+1}^{-1}} \otimes \dots \otimes a_{\pi_{i+j}^{-1}},$$

i.e. $a_1 \otimes \dots \otimes a_i$ and $a_{i+1} \otimes \dots \otimes a_{i+j}$ are subsequences of $a_{\pi_1^{-1}} \otimes \dots \otimes a_{\pi_{i+j}^{-1}}$.

In our case of an exterior algebra, let $\text{char } K = 2$, so that A is commutative. Then $HH_*(A; A)$ and $HH^*(A; A)$ are commutative graded A -algebras. Since all above isomorphisms are A -linear, a similar argument using 1.5 shows that

$$HH_*(A; A) \cong HH^*(A; A) \cong A[y_1, \dots, y_n].$$

5. AFTERWORD

Given a simplicial complex Δ with vertex set $[n]$, the Stanley-Reisner algebra (SRa) of Δ is the monomial quotient $\Lambda[x_1, \dots, x_n \mid x_{\sigma}; \sigma \notin \Delta]$. Its module basis consists of all the simplices of Δ . In the above statements, we have computed HH of the SRa of an n -ball. It is an interesting task to determine HH of the SRa of a sphere $\Lambda[x_1, \dots, x_n \mid x_1 \cdots x_n]$ or a path $\Lambda[x_1, \dots, x_n \mid x_i x_j; |i-j| \geq 2]$ or a cycle $\Lambda[x_1, \dots, x_n \mid x_i x_j; |i-j \bmod n| \geq 2]$ or other more general simplicial complexes. This still appears to be an open problem.

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